Probing Students' Levels of Geometric Thinking in Geometry and Their Enacted Example Space Function

Robin B. Dimla *

Abstract

The study explored the pre-service secondary Mathematics teachers' levels of geometric thinking in geometry and their enacted example space function while they were exposed to van Hiele model instruction. The findings of the study reveal that most of the students were functioning at the recognition level in plane geometry and the highest geometric thinking level manifested prior to their exposure to van Hiele model is the informal deduction level. The evidences based on the study show that students' example space function from various phases of instruction does not depend on their levels of geometric thinking that they had in plane geometry. Students with various levels of geometric thinking were able to generate quality examples across phases of instruction. Evidences likewise support the claim that van Hiele model assists students' development of their example space function. Prior to the instruction misconception on properties of prism was evident among students. After the instruction, the breadth and the depth of understanding in relation to properties of prism was evident based on the quality of examples they provided. The most dominant geometric thinking level after students' exposure to the van Hiele model instructi on is also the recognition level.

Keywords: Geometric thinking, Example space function, van Hiele Model, Plane and Solid Geometry

1. Introduction

Geometry is an integral part of the mathematics curriculum because of its application in real -life situations. The study of geometry helps students to develop their skills in visualization, critical thinking, intuition, problem solving, conjecturing, deductive reasoning, logical argument and proof (Jones, 2000). Research shows that many students who have studied ge ometry formally did not develop logico-mathematical reasoning and are still at most at level 2 (Fuys, et.al, 1988). The researcher has observed that students who are studying formal proofs frequently encounter problems on the logical structure of statements. They fail to establish possible connection between statements. They know how to start by simply determining the hypothesis but the succeeding statements are apparently treated as complex and complicated. Thus, the skill is very much limited and seeming ly fails to lead at a logically structured proof. The development of learner's deductive reasoning is an aim of geometry. Hence, the investigation of geometry teaching as an axiomatic deductive mathematical system is highly warranted in order to bridge the possible gap.

Geometry has presentation in two and three dimensions. Plane geometry deals with the study of flat surfaces which has two dimensions, while solid geometry deals with geometric shapes having three dimensions. A sound and good knowledge in plane geometry is necessary in the study of solid geometry. In geometry, one of the most well -defined models for student's level of thinking is Van Hiele's model. The Van Hiele's model has five levels, namely: visualization, analysis, informal deduction, formal deduction, and rigor (Clements & Battista, 1990). On the first level (visualization), a student recognizes geometric shapes. On the

* Assistant Prof. Dr., Don Honorio Ventura Technological State University, Philippines
E-mail: rbdimla319@dhvtsu.edu.ph

second level (analysis), a student identifies the properties of certain shapes. On the third level (informal deduction), a student comprehends the relation between shapes and creates the relationships. On the fourth level (formal deduction), a student appreciates the meaning and importance of deduction by making use of postulates, theorems, and proofs. On the fifth level (rigor), a student makes more abstractions. The theory also posits that a student has to pass through the lower levels before he gets to the higher and highest levels.

The Van Hiele model proposes learning phases that assist students to move from a lower to a higher level of geometric thinking. These phases are information, guided orientation, explication, free orientation, and integration. In the Information phase, the interaction between teacher and student through discussion is emphasized. Students make discoveries using guided activity in the guided orientation phase. In the explication phase, students can explain and express their views about the observed structure. Students can explain more complex tasks in the free orientation phase. In the integration phase, a student summarizes the lesson learned for the purpose of establishing a new overall view (Crowley, 1987). Difficulties in teaching and learning geometry have been pointed out by numerous researchers. Teaching a geometry lesson at one Van Hiele level when students are functioning at a lower level may hinder students' learning (Gutiérrez, 1992.). Students who are at lower levels of thinking cannot be expected to understand instructions presented to them at a higher level of thinking. This is because each level of thinking has its own language (Teppo, 1991). A teacher should design tasks and activities that are in line with the students' levels of thinking (Pegg & Davey, 2012). In order to succeed in moving the students from the lower level to the higher levels, more sophisticated tasks and/or activities should be introduced (Siyepu, 2005).

In teaching topics in mathematics, emphasis in the development of concepts and skills is increased through the use of examples (Zazkis & Leikin, 2008). Students' understanding of mathematical concepts is mirrored by their generated or enacted examples. Example spaces are the examples learners produce that arise from a small pool of ideas that simply appear in response to particular tasks in particular situations (Watson & Mason, 2005). In other words, enacted examples are those created by the students which are the indicators of their knowledge and understanding. Examples are used to develop intuition and as means of generating, testing, and refining conjectures (Alcock & Inglis, 2008). The purpose of the example is to provide a more familiar and concrete means to explore ideas and to check the conditions of and evaluate the constraints in theorem formulation (Fukawa-Connelly & Newton, 2014). Examples can play a critical role in the exploration of conjectures and in the subsequent development of proofs (Lockwood et al., 2016).

Studies on the development of geometric thinking in plane and solid geometry with the integration of enacted example space of students are rare. For this reason, an exploratory study on the students' level of geometric thinking and their enacted example space in different learning phases in solid geometry is timely and relevant. Many students dislike and feel uncomfortable in learning geometry. Identifying the levels of geometric thinking of students in plane geometry and their enacted example space function in different learning phases in solid geometry may provide insights and bases in proposing a model of teaching solid geometry.

2. Method

2.1. Research Design

The study employed a descriptive research design in exploring students' levels of geometric thinking and their enacted example space function. In particular, the study described the students' levels of geometric thinking in plane geometry prior to instruction in solid geometry, explored students' enacted example space function in the different phases of instruction in the van Hiele model, described students' levels of geometric thinking in solid geometry, and proposed a sample of teaching guides in solid geometry.

This study was conducted at the Don Honorio Ventura Technological State University (DHVTSU), in Bacolor, Pampanga. The location of the school is dubbed as the Athens of Pampanga for being the proud home of poets, artists and the learned.

The respondents of the study were 35 students (one intact class) of Bachelor of Secondary Education major in Mathematics at DHVTSU who are enrolled in Mathematics 313b (Solid Geometry) during the first semester of school year 2016-2017. They were purposively selected to compose the class who were provided instruction in solid geometry. The respondents have taken the Plane Geometry course during the previous semester, a pre-requisite subject for Solid Geometry.

Four (4) instruments were used in gathering the data needed in the study. These include the van Hiele Geometry Test, example log, geometric thinking test in Solid Geometry and an observation protocol.

The researcher used a validated geometry test called Van Hiele Geometry Test (VHGT) designed by Usiskin (1982) to measure the students' level of geometric thinking in plane geometry. The test consists of 25 multiple-choice geometry questions. An example log was used to record students' generated examples in order to describe students' enacted example space function in the different learning phases in solid geometry. The example log includes listings of examples from the different phases of instruction, phase of learning where the example occurs and the quality of the example. The geometric thinking test in solid geometry was used to assess the students' level of geometric thinking in solid geometry. The test consists of twenty five (25) items and was administered after the eight-week instruction following the Van Hiele model. An observation protocol instrument was used by three mathematics teachers at DHVTSU to observe class instruction in solid geometry, particularly the learning phases undertaken. The observers assessed the instruction along the information phase, guided orientation phase, explication phase, free orientation phase and integration phase.

2.2. Analysis of Data

Students' levels of geometric thinking in plane geometry prior to instruction in solid geometry were described based on the scores obtained in the Van Hiele geometry test which were classified using Usiskin's grading system.

A student was given or assigned a weighted sum score in the following manner: (a) 1 point for meeting the criteria on items 1-5 (level-I); (b) 2 points for meeting the criteria on items 6-10 (level-II); (c) 4 points for meeting the criteria on items 11-15 (level-III); (d) 8 points for meeting the criteria on items 16-20 (level-IV), and (d) 16 points for meeting the criteria on items 21-25 (level-V). By using the classical, modified and forced Van Hiele levels for classification, almost every student can be assigned to a geometric level. If almost all the students have a Van Hiele level, the data become easier for analysis.

The weighted sum score for the classical Van Hiele level, the modified Van Hiele level and the forced van Hiele level are adopted from Usiskin (1982). The qualitative analysis for students' levels of geometric thinking in plane geometry was likewise performed based on the specific responses of the students on sample items taken from the test.

To explore students enacted example space function in the different phases of instruction in solid geometry, analysis of students' example space from the different phases of instruction is based on the correctness of generated examples (Zazkis & Leikin, 2008). Correctness is evaluated based on the appropriateness of the example. Appropriateness is classified as correct logical structure of the statements or appropriate with limited structure. Correct logical structure means that the generated example is necessary and sufficient. It becomes necessary and sufficient if these examples are based on expert example spaces. Limited in structure means that the example is taken from the expert example space, however, some conditions are missing to specifically differentiate the figure from other figures.

Expert example spaces include the rich variety of expert knowledge as well as the instructional example spaces. Instructional example spaces include examples generally understood by mathematicians and are displayed in textbooks and most often used in instruction rigor.

3. Results and Discussion

3.1. Students' Levels of Geometric Thinking in Plane Geometry

Figure 1 shows the students' levels of geometric thinking in plane geometry prior to the instruction in solid geometry based on the Van Hiele Geometry Test (VHGT).

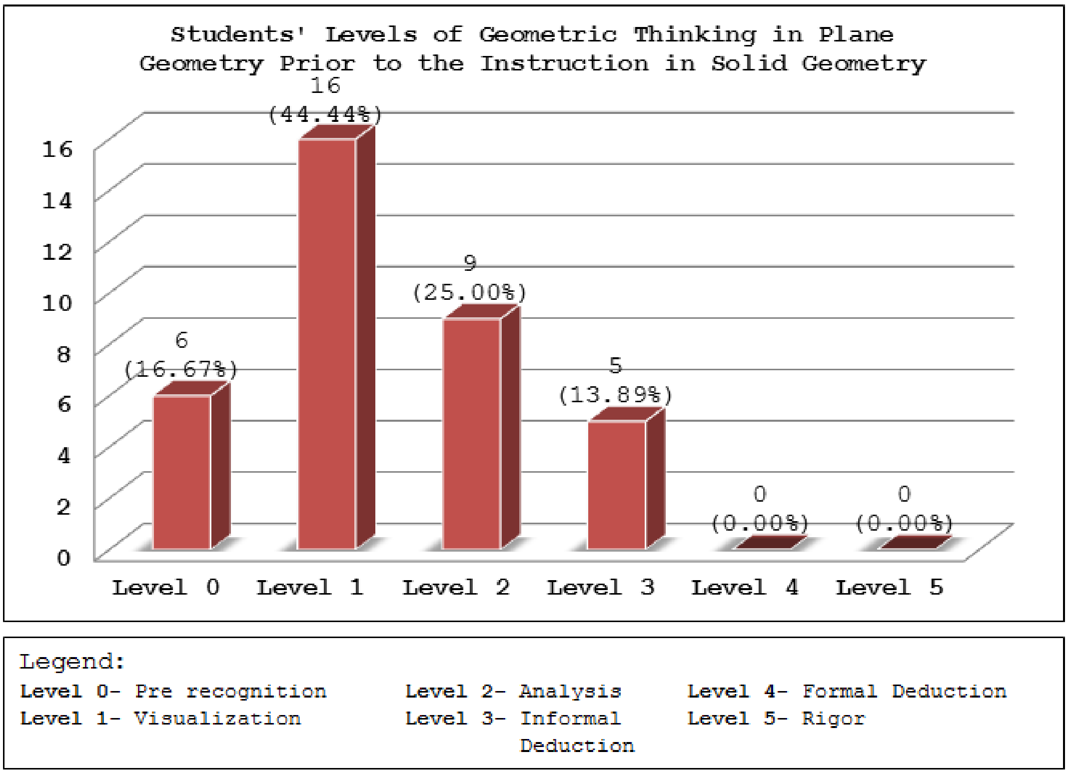

Figure 1. Students Levels of Geometric Thinking in Plane Geometry prior to Instruction in Solid Geometry

Most of the students were functioning at Van Hiele level 1 or the visualization level in plane geometry, i.e., 16 out of 36 or 44.44% of the students. The evidences from the study showed that students at this level recognized figures by their physical appearance; however, they have difficulties in describing shapes on the basis of their properties.

Nine (9) or 25% of the students were at the analysis level. Findings reveal that these students have difficulty in forming correct informal deductive arguments and implicitly using such logical forms.

Five (5) or 13.89% of the students reached level 3 or the informal deduction level. This level was the highest classification reached by the students. Findings were consistent with the related studies that students at the informal deduction level recognize the relationships between and among properties of figures. There are also evidences of logical ordering of the properties of shapes.

Considering the fact that these students have a one-semester study of plane geometry, none of them were classified at level 4 and level 5 and 6 out of 36 or 16.67% of the students were functioning at level 0 or the pre-recognition level.

Evidences of the present study support the findings of related studies (Knight, 2006; Meng & Sam, 2009; Wang, 2014) on Preservice Secondary Mathematics Teachers' geometric thinking. i.e., informal deduction level appeared to be the highest level achieved by the students. In all of these studies, none of the pre-service secondary mathematics teachers showed a level 5 (rigor) reasoning stage in geometry. Properties expected at each level based on the findings are likewise consistent with the descriptions of Crowley (1987) and Clements and Battista (1990). Students' ability to deduce properties and recognize classes of figures were observed at higher level of geometric thinking. Also, difficulty in visualizing figures on the basis of their properties was observed at lower level of geometric thinking.

3.2. 3.2 Students' Example Space Function from the Different Phases of Instruction

At the information phase of the visualization level in lesson 1 (Properties of prisms, lateral and total areas of prisms), students functioning at the pre-recognition level, visualization level, analysis level and informal deduction level provided a combination of appropriate and inappropriate examples. Of these appropriate examples, all have limited structure. Limited structure suggests that generated examples are necessary but cannot be distinctively differentiated from other figures considering these properties. In lesson 2 (volume of prisms), even though all examples provided by the students from different levels of geometric thinking are appropriate, they are only limited on space, the unit and the volume of rectangular prism.

At the guided orientation phase of the visualization level in lesson 1, students functioning at different levels of geometric thinking registered high level of appropriateness in terms of the solid figures they classified as prisms. This suggests that most of the objects were classified correctly by the students as prism and non-prism on the basis of their appearance.

Most of the examples provided by the students functioning at different levels of geometric thinking are appropriate with correct logical structure which suggests that they are both necessary and sufficient.

At the analysis level, properties discovered by the students regardless of levels of geometric thinking are appropriate in terms of the structure. Only students functioning at the visualization level noticed that in a case of a cube, lateral faces and bases are congruent. In lesson 2, based on the mean rating of the students functioning from different levels of geometric thinking, the level of appropriateness was described to be high. This means that most of the objects were correctly constructed by the students and number of cubes were correctly identified. However, students functioning at the pre-recognition level registered the lowest mean of 3.60 and students functioning at the analysis level got the highest mean of 4.63.

At the explication phase of the visualization level in lesson 1, students from the pre-recognition level to informal deduction level appropriately identified various parts of the prisms based on the properties taken from the expert's example space. Their exposure from various activities in the previous phase assisted them. This suggests that regardless of the levels of geometric thinking, students' example space function may be developed based on the quality of the activities provided among the students. At the analysis level, students from various levels of geometric thinking were able to discover that the lateral area of a prism can be associated with the area of a rectangle. Therefore, the sum of the base edges can be associated with the length of the rectangle and the lateral edge is for the width of the rectangle. However, this property is only true for right prisms. Only students at the analysis level were able to discover that for oblique prisms, lateral area is the same as the area of the parallelogram. In lesson 2, students from different levels of geometric thinking had complete understanding that volume of a solid is the number of times a unit of solid is contained in the given solid like the example provided by the students that the volume of a box is the number of cubic inches in the box, the volume of a room is the number of cubic feet contained in the room.

At the free orientation phase of the visualization level in lesson 1, regardless of the students' levels of geometric thinking, net construction is based on the properties they got from the previous phase. Students were able to construct the nets of trapezoidal, dodecagonal and octagonal prisms. From this activity, students from various levels of geometric thinking appropriately constructed the nets of the prism based on logical sequence of the process performed. Logical sequence means understanding or knowing what has to be the foremost and the subsequent among the steps of the construction. From this activity, they started with the construction of the base leading to the other parts of the prism.

At the informal deduction level, most of the students' responses from various levels of geometric thinking reveal some evidences of understanding proof. Properties explored from the previous phase were used to justify statements. However, the structure of the proof was guided following the questions provided among the students. Students used direct proof starting with the hypothesis followed by sequentially formulated statements leading to the conclusion. In lesson 2, students functioning at different levels of geometric thinking observed that the two sets of objects differ only in arrangement but they have the same height, bases and volume based on the number of coins for the two sets.

At the integration phase of the visualization level in lesson 1, common realizations were observed among the students at the pre-recognition and visualization level. Going through the phases of instruction made them realize that prisms differ in terms of its base. The base can be a regular polygon or not. Congruent lateral faces are for those prisms with whose base is a regular polygon. Students at the analysis level discovered the similarities and differences of each prism. Also, they found out that lateral faces are based on the measurement of the base. Students realized that there are many types of prisms beyond the 2 prisms that they know. Lateral faces in connection with the base of the prism were also observed by the students at the informal deduction level. They also added the many types of prisms that they discovered after going through the activities.

At the informal deduction level, evidences showed that regardless of the students' levels of geometric thinking the activities provided from the different phases of learning assisted them to discover properties of prism beyond the usual properties that the students know. Mastery of the said properties was evident among students from the pre-recognition level to informal deduction level. In lesson 2, regardless of the students' levels of geometric thinking, they showed complete understanding of the Cavalieri's theorem that "If two solids are included between two parallel planes, and if the two sections cut from them by any plane parallel to the including planes are equal in area, the volumes of the solids are equal". Also, students have a clear understanding of the volume of prism, i.e., the product of the area of its base and altitude.

3.3. Students' Geometric Thinking in Solid Geometry

After the instruction in solid geometry designed based on the *van Hiele model*, students' levels of geometric thinking were assessed. The result is presented in Figure 2.

The most dominant level of geometric thinking in solid geometry is the visualization level (i.e., 13 out of 36 or 36.11% of students). Students at this level have already shown some manifestations of moving towards the analysis level. They have recognized figu res beyond the physical appearance.

There were undergraduate students who were able to attain the higher levels of geometric thinking, i.e., the formal deduction level and rigor. However, six (6) out of 36 or 16.67% of the students were fuctioning at level 0 or the pre -recognition level.

The number of students functioning at the recognition level dropped by three students after the instruction. Students at the analysis level decreased by five. Instruction based on the Van Hiele model increased the number of students functioning at the informal deduction level from five students prior to the instruction to 11 students after the instruction. According to the Van Hiele model, the five phases of instruction support students as they progress through the levels of geometric thinking.

Contrary to the findings of related studies (Knight, 2006; Meng & Sam 2009; Wang, 2009) there are undergradute students who were able to attain the higher levels of geometric thinking, i.e., the formal deduction level and rigor. Evidences likewise support the characteristics of the students at each level of geometric thinking identified by Crowley (1987) and Gutierrez (1992).

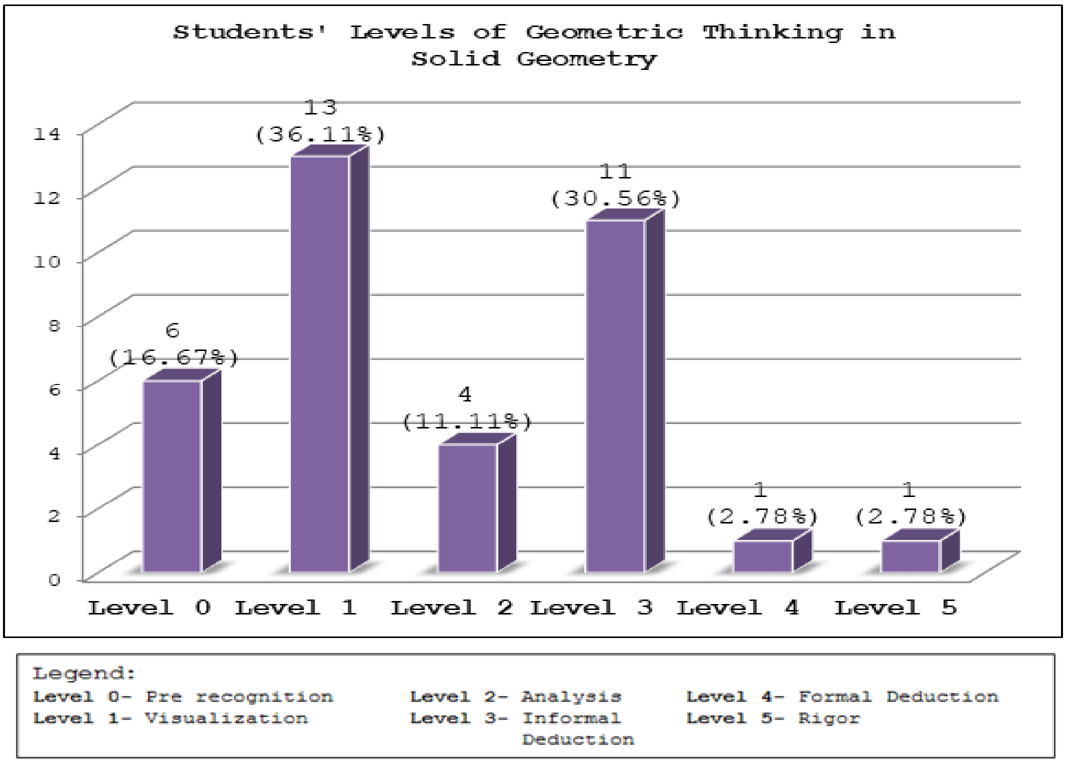

Figure 2. Students' Levels of Geomtric Thinking in Solid Geometry

4. Conclusions

The following conclusions were drawn:

1. In plane geometry, most of the students are functioning at lower level of geometric thinking. The highest classification reached by the students is the informal deduction level.
2. The students' example space function from various phases of instruction does not depend on their levels of geometric thinking that they had in plane geometry.
3. Most of the students are also functioning at a lower level of geometric thinking in solid geometry. However, there are students who were able to achieve higher levels of geometric thinking, the formal deduction level and rigor, in solid geometry.

5. Recommendations

The findings lead to the following recommendations:

1) Professors / instructors in plane geometry may revisit their instruction emphasizing more opportunities for their students to develop higher levels of geometric thinking. They should revise their instructional methods to utilize the van Hiele' strategies in planning and delivering lessons.

2) Professors / instructors are encouraged to take some professional training course that is related to van Hiele model which provides not only an opportunity for them to improve their insight of geometry but also helps them understand the van Hiele model which can directly influence their way of pedagogy in geometry.

3) Understanding students' levels of geometric thinking can be integrated on assessment of students' learning outcome so that teachers in geometry are guided on how their students are performing in their subject.

4) Solid geometry may be taught following the van Hiele model since evidences support its effectiveness in achieving higher levels of geometric thinking and developing students' example space function.

5) Curriculum planners may consider the inclusion of van Hiele model in all courses in geometry following the phases of instruction.

6) More in depth study may be conducted to explore more evidences that students at the undergraduate level with a richer example space are more able to abstract, generalize and write proofs.

7) Researchers are encouraged to make further study along this line to refute or affirm the findings.

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